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AN ALGORITHM FOR EFFICIENT ESTIMATION OF SUPERIMPOSED EXPONENTIAL SIGNALS

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ABSTRACT

A computational algorithm is given for obtaining asymptotically efficient estimates of the unknown complex amplitudes and frequencies in a superimposed exponential model for signals. It is shown that the variance covariance matrix of these estimates are asymptotically the same as that for the maximum likelihood estimates and thus attain the Cramér-Rao lower bound.

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Key Words and Phrases: Cramér-Rao lower bound, Equivariation linear prediction, Forward and backward linear prediction, Maximum likelihood estimate, Superimposed exponential signals.

1. INTRODUCTION

We consider the superimposed exponential signals model

$$y_t = \alpha_1 e^{i\omega_1 t} + \dots + \alpha_p e^{i\omega_p t} + \epsilon_t, \quad t = 1,\dots,N \tag{1.1}$$

where $i = \sqrt{-1}$, α_s is a complex amplitude, ω_s is the frequency of the s-th signal and $\{\epsilon_t\}$ is a sequence of iid complex random variables such that

$$E(\epsilon_t) = 0,\ E(re\ \epsilon_t)^2 = E(im\ \epsilon_t)^2 = 2^{-1}\sigma^2$$
$$\operatorname{cov}(re\ \epsilon_t,\ im\ \epsilon_t) = 0 \tag{1.2}$$

where *re* and *im* indicate real and imaginary parts of a complex number.

The problems of interest are the estimation of the unknown frequencies $\omega_1,\dots\omega_p$ assumed to be different, and the estimation of the unknown complex amplitudes $\alpha_1,\dots,\alpha_p$.

When ϵ_t has a complex normal distribution, the MLE's (maximum likelihood estimators) of the unknown parameters are the same as the non-linear LSE's (least squares estimators) obtained by minimizing

$$\sum_{t=1}^{N} |y_t - \sum_{s=1}^{p} \alpha_s e^{i\omega_s t}|^2 \tag{1.3}$$

with respect to α_s and ω_s, $s = 1,\dots,p$. Unfortunately, there is no closed form solution to this problem. In this paper, we develop a

computer algorithm which will provide estimates which are asymptotically as efficient as the MLE's.

Let us write the unknown parameters

$$re\ \underset{\sim}{\alpha} = (re\ \alpha_1, \ldots, re\ \alpha_p)' \tag{1.4}$$

$$im\ \underset{\sim}{\alpha} = (im\ \alpha_1, \ldots, im\ \alpha_p)' \tag{1.5}$$

$$\underset{\sim}{\omega} = (\omega_1, \ldots, \omega_p)' \tag{1.6}$$

and denote the Fisher information matrix for all the parameters (1.4)-(1.6) by F_N. Further let

$$T_N = \mathrm{diag}(N^{1/2} I_p,\ N^{1/2} I_p,\ N^{3/2} I_p)$$

where I_p is the identity matrix of order p and define $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_p)$. Then we have

$$\lim_{N\to\infty} T_N^{-1} F_N T_N^{-1} = \sigma^{-2} \begin{pmatrix} 2I_p & 0 & -im\ A \\ 0 & 2I_p & re\ A \\ -im\ A & re\ A & \frac{2}{3} A^* A \end{pmatrix}. \tag{1.7}$$

From (1.7), we expect that $\hat{\underset{\sim}{\omega}}$ and $\hat{\underset{\sim}{\alpha}} = (re\ \hat{\underset{\sim}{\alpha}},\ im\ \hat{\alpha})'$ the MLE's of $\underset{\sim}{\omega}$ and $\underset{\sim}{\alpha}$ have the limiting distributions

$$N^{3/2}(\hat{\underset{\sim}{\omega}} - \underset{\sim}{\omega}) \longrightarrow N_p(0, 6\sigma^2 (A^* A)^{-1}) \tag{1.8}$$

$$N^{1/2}(\hat{\underset{\sim}{\alpha}} - \underset{\sim}{\alpha}) \longrightarrow N_{2p}(0, \sigma^2 V) \tag{1.9}$$

where

$$V = \begin{pmatrix} \frac{1}{2}I_p+\frac{3}{2}(im\ A)^2(A^*A)^{-1} & -\frac{3}{2}(re\ A\ im\ A)(A^*A)^{-1} \\ -\frac{3}{2}(re\ A\ im\ A)(A^*A)^{-1} & \frac{1}{2}I+\frac{3}{2}(re\ A)^2(A^*A)^{-1} \end{pmatrix}$$

We show that the estimators we propose have the same limiting distributions as (1.8) and (1.9) and are thus asymptotically efficient, i.e., attain the lower bounds for their asymptotic covariances.

The best known methods of estimation for the frequencies $\omega_1,\ldots,\omega_s$, like the modified FBLP (forward and backward linear prediction) of Tufts and Kumaresan (1982) and EVLP (equivariation linear prediction) discussed in Bai, Krishnaiah and Zhao (1986) and Rao (1988) have certain deficiencies. The modified FBLP estimates are not consistent, although simulation results support their validity in small samples when the SNR (signal to noise ratio) is relatively low. The EVLP provides estimates of the frequencies which are asymptotically normal and have a convergence rate of $O(N^{-1/2})$. However, this is still not the best possible. In the next section, we describe the EVLP method and show how the ELVP estimates could be refined to produce fully efficient estimates of the frequencies $\omega_1,\ldots,\omega_s$ with the best possible convergence rate of $O(N^{-3/2})$.

2. THE EVLP METHOD

Suppose that the vector $\underset{\sim}{b} = (b_0,b_1,\ldots,b_p)'$ is such that

$$b_0 + b_1z + \ldots + b_pz^p = b_p \prod_{s=1}^{p} (z-e^{-i\omega_s}). \qquad (2.1)$$

Then for any $n \geq p+1$

$$\sum_{k=0}^{p} b_k y_{n-k} = \sum_{k=0}^{p} b_k \epsilon_{n-k} \tag{2.2}$$

where the right hand side of (2.2) is a function of error only. The coefficients b_i are estimated by minimizing

$$\sum_{n=p+1}^{N} \left| \sum_{k=0}^{p} b_k y_{n-k} \right|^2 \tag{2.3}$$

subject to the conditions $b_0 > 0$ and $|\underset{\sim}{b}| = 1$. Such a method of estimation is known as the EVLP method. It may be noted that in the LP and FBLP methods, the expression (2.3) is minimized subject to the condition that $b_0 = 1$. [Unfortunately, the restriction $b_0 = 1$ is not sufficient to ensure the consistency of the estimates of the ratios of the b_i coefficients.]

Now write

$$\hat{\gamma}_{rs} = \frac{1}{N-p} \sum_{t=p+1}^{N} \bar{y}_{t-r} y_{t-s}, \quad r,s = 0,1,\ldots,p \tag{2.4}$$

and construct the $(p+1)\times(p+1)$ matrix

$$\hat{\Gamma} = (\hat{\gamma}_{rs}). \tag{2.5}$$

It is easily seen that the EVLP estimate $\hat{\underset{\sim}{b}}$ of $\underset{\sim}{b}$ is the unit eigen vector with a non-negative first element providing the smallest eigen value of $\hat{\Gamma}$. We use $\hat{\underset{\sim}{b}}$ to construct the polynomial equation

$$\hat{b}_0 + b_1 z + \ldots + \hat{b}_p z^p = 0, \tag{2.6}$$

obtain solutions in the form

$$\tilde{\rho}_1 e^{-i\tilde{\omega}_1}, \ \ldots, \ \tilde{\rho}_p e^{-i\tilde{\omega}_p} \tag{2.7}$$

and take $\tilde{\omega}_1,\ldots,\tilde{\omega}_p$ as estimates of $\omega_1,\ldots,\omega_p$. It is shown in Bai, Krishnaiah and Zhao (1986) that $\tilde{\underset{\sim}{\omega}}$ is a consistent estimate of $\underset{\sim}{\omega}$ with a convergence rate of $O_p(N^{-1/2})$.

3. THE MAIN THEOREM

Let $\tilde{\omega}_s$ be an estimate of ω_s, $s = 1,\ldots,p$ and compute

$$\hat{\omega}_s = \tilde{\omega}_s + \frac{12}{N^2}\, im\left(\frac{C_N}{D_N}\right) \tag{3.1}$$

where

$$C_N = \sum_{t=1}^{N} y_t\left(t - \frac{N}{2}\right)e^{-i\tilde{\omega}_s t} \quad \text{and} \quad D_n = \sum_{t=1}^{N} y_t e^{-i\tilde{\omega}_s t}.$$

Then we have the following theorem.

<u>Theorem</u>. Suppose that ϵ_t satisfies the conditions (1.2), ω_s, $s = 1,\ldots,p$, are distinct, and

$$\tilde{\omega}_s-\omega_s = O_p(N^{-1-\delta}), \quad \delta \in (0,\tfrac{1}{2}], \quad s = 1,\ldots,p. \tag{3.2}$$

Then

$$\text{(i)} \quad \hat{\omega}_s-\omega_s = O_p(N^{-1-2\delta}) \quad \text{if} \quad \delta \le \tfrac{1}{4} \tag{3.3}$$

$$\text{(ii)} \quad N^{3/2}(\hat{\underset{\sim}{\omega}}-\underset{\sim}{\omega}) \longrightarrow N_p(0,6\sigma^2(A^*A)^{-1}) \quad \text{if} \quad \delta > \tfrac{1}{4} \tag{3.4}$$

where $A = \text{diag}(\alpha_1,\ldots,\alpha_p)$.

In the next section, we show that starting with the EVLP estimates $\tilde{\omega}_s$ of ω_s and using the formula (3.1) repeatedly, we arrive at fully efficient estimates of ω_s having the limiting distribution (3.4).

<u>Proof</u>. We have

$$\begin{aligned}\sum_{t=1}^{N} y_t e^{-i\tilde{\omega}_s t} &= \sum_{m=1}^{p} \alpha_m \sum_{t=1}^{N} e^{i(\omega_m-\tilde{\omega}_s)t} + \sum_{t=1}^{N} \epsilon_t e^{-i\tilde{\omega}_s t} \\ &\triangleq \sum_{m=1}^{p} \alpha_m J_m(N) + R(N). \end{aligned}\tag{3.5}$$

It is easy to see that

$$\begin{aligned} J_m(N) &= \begin{cases} O_p(1) & \text{if} \quad m \ne s \\ N+i(\omega_s-\tilde{\omega}_s) \sum_{t=1}^{N} t\ e^{i(\omega_s-\omega_s^*)n} & \end{cases} \\ &= N + O_p(N^{1-\delta}) \quad \text{if} \quad m = s \end{aligned}\tag{3.6}$$

where $\omega_s^* \in (\omega_s, \tilde{\omega}_s)$, and

$$R(N) = \sum_{t=1}^{N} \epsilon_t e^{-i\tilde{\omega}_s t}$$

$$= \sum_{t=1}^{N} \epsilon_t e^{-i\omega_s t} + \sum_{j=1}^{L-1} \frac{[-i(\tilde{\omega}_s - \omega_s)]^j}{j!} \sum_{t=1}^{N} \epsilon_t t^j e^{-i\omega_s t}$$

$$+ \frac{\theta[N(\tilde{\omega}_s - \omega_s)]^L}{L!} \sum_{t=1}^{N} |\epsilon_t| \tag{3.7}$$

where $|\theta| \leq 1$ and $L\,\delta > 1$. From (3.7) computing the order of the terms on the right hand side, we have

$$R(N) = O_p(N^{1/2}) + \sum_{j=1}^{L-1} \frac{O_p(N^{-(1+\delta)j})}{j!} O_p(N^{j+1/2}) + O_p(1) = O_p(N^{1/2}). \tag{3.8}$$

The expressions (3.5)-(3.8) imply that

$$\sum_{t=1}^{N} y_t e^{-i\tilde{\omega}_s t} = \alpha_s N(1 + O_p(N^{-\delta})). \tag{3.9}$$

Similarly, one can prove

$$\sum_{t=1}^{N} y_t (t - \frac{N}{2}) e^{-i\tilde{\omega}_s t} = \sum_{t=1}^{N} \epsilon_t (t - \frac{N}{2}) e^{-i\omega_s t} - i\alpha_s (\frac{N^3}{12}(1 + O_p(N^{-\delta}))(\tilde{\omega}_s - \omega_s). \tag{3.10}$$

By (3.9) and (3.10) we obtain

$$\hat{\omega}_s = \tilde{\omega}_s + \frac{12}{N^2}\, im\, \frac{\sum_{t=1}^{N} \epsilon_t (t-\frac{N}{2}) e^{-\omega_s t} - i\alpha_s (\frac{N^3}{12}(1+O_p(A)^{-\delta}))(\tilde{\omega}_s - \omega_s)}{\alpha_s N(1+O_p(N^{-\delta})}$$

$$= \omega_s + O_p(N^{-\delta})(\tilde{\omega}_s - \omega_s) + \frac{12}{N^3}\, im(\alpha_s^{-1} \sum_{t=1}^{N} \epsilon_t (t - \frac{N}{2}) e^{-i\omega_s t}). \tag{3.11}$$

Then the theorem follows from (3.11) using the following fact

$$\{(\frac{N^3}{12})^{-1/2} \sum_{t=1}^{N} \epsilon_t (t - \frac{N}{2}) e^{-i\omega_s t}, \; s = 1,2,\ldots,p\} \longrightarrow CN(0,\sigma^2 I_p). \tag{3.12}$$

4. RECURSIVE ALGORITHM FOR ESTIMATION

We start with a consistent estimate of ω_s and improve upon it step by step by a recursive algorithm. The m-th stage estimate $\hat{\omega}_s^{(m)}$ is computed from the (m-1)th stage estimate by the formula

$$\hat{\omega}_s^{(m)} = \hat{\omega}_s^{(m-1)} + \frac{12}{N_m^2}\, im\left(\frac{C_m}{D_m}\right), \; m = 1,2,\ldots \tag{4.1}$$

where

$$D_m = \sum_{t=1}^{N_m} y_t e^{-i\hat{\omega}_s^{(m-1)} t} \tag{4.2}$$

$$C_m = \sum_{t=1}^{N_m} y_t (t - \frac{N_m}{2}) e^{-i\hat{\omega}_s^{(m-1)} t}. \tag{4.3}$$

We apply the formula (4.1) repeatedly choosing N_m suitably at each stage.

<u>Step 1 with m = 1</u>. Choose $N_1 = [N^{0.4}]$ and $\hat{\omega}_s^{(0)} = \tilde{\omega}_s$ the EVLP estimate. Note that

$$\tilde{\omega}_s - \omega_s = O_p(N^{-1/2}) = O_p[N_1^{-1-1/4}]. \tag{4.4}$$

Then substituting $N_1 = [N^{0.4}]$ and $\hat{\omega}_s^{(0)} = \tilde{\omega}_s$ in the formula (4.1) we find by applying the result (3.3) of the main theorem

$$\hat{\omega}_s^{(1)} - \omega_s = O_p(N_1^{-1-1/2}) = O_p(N^{-0.6}). \tag{4.5}$$

<u>Step 2 with m = 2</u>. Choose $N_2 = [N^{0.48}]$ and using $\hat{\omega}_s^{(1)} - \omega_s = O_p(N^{-0.6})$ $= O_p(N_2^{-1-1/4})$, compute $\hat{\omega}_s^{(2)}$ by the formula (4.1). Again by the result (3.3) of the main theorem

$$\hat{\omega}_s^{(2)} - \omega_s = O_p(N_2^{-1-1/2}) = O_p(N^{-0.72}). \tag{4.6}$$

<u>Steps 3 to 7</u>. Choosing $N_3,\ldots,N_7$ as given below and applying the main theorem in the same way, we have

$$N_3 = [N^{0.57}] \quad \text{yielding} \quad \hat{\omega}_s^{(3)} - \omega_s = O_p(N^{-0.84})$$

$$N_4 = [N^{0.67}] \quad \text{"} \quad \hat{\omega}_s^{(4)} - \omega_s = O_p(N^{-1})$$

$$N_5 = [N^{0.80}] \quad \text{"} \quad \hat{\omega}_s^{(5)} - \omega_s = O_p(N^{-1.20})$$

$$N_6 = [N^{0.96}] \quad \text{"} \quad \hat{\omega}_s^{(6)} - \omega_s = O_p(N^{-1.44}). \tag{4.7}$$

Finally we take $N_7 = N$ and compute $\hat{\omega}_s^{(7)}$. Now applying the result (3.4) of the main theorem, we have

$$N^{3/2}(\hat{\underset{\sim}{\omega}}^{(7)} - \underset{\sim}{\omega}) \longrightarrow N(\underset{\sim}{0}, 6\sigma^2(A^*A)^{-1}) \tag{4.8}$$

which is the same as that of the MLE given in (1.8) and shows that $\hat{\underset{\sim}{\omega}}^{(7)}$ is a fully asymptotically efficient estimate of $\underset{\sim}{\omega}$.

Using the estimate $\hat{\underset{\sim}{\omega}}^{(7)}$ of $\underset{\sim}{\omega}$, $\underset{\sim}{\alpha}$ is estimated by

$$\hat{\alpha} = (\hat{\Omega}^*\hat{\Omega})^{-1}\hat{\Omega}^*\underset{\sim}{y} \tag{4.9}$$

where $\hat{\Omega}$ is a $n \times p$ matrix whose (r,s)-th element is $\exp(ir\hat{\omega}_s^{(7)})$. It is easily shown that $\hat{\alpha}$ has the same limiting distribution as that of the MLE given in (1.9).

<u>Remark 1</u>. Note that $\hat{\Omega}^*\Omega = N(I+O_p(N^{-1}))$ so that (4.9) can be more simply approximated by

$$\hat{\alpha} = N^{-1}\Omega^{*}\underset{\sim}{y}. \tag{4.10}$$

Remark 2. The recursive formula (4.1) for N_m can be improved by using the alternative formula

$$\hat{\omega}_s^{(m)} = \hat{\omega}_s^{(m-1)} + \frac{12}{N_m^2(N-N_m+1)} \sum_{t=0}^{N-N_m} im\, \frac{C^*_{mt}}{D^*_{mt}} \tag{4.11}$$

where

$$C^*_{mt} = \sum_{n=1}^{N_m} y_{n+t}(n - \frac{N_m}{2})e^{-i\hat{\omega}_{m-1}n}$$

$$D^*_{mt} = \sum_{n=1}^{N_m} y_{n+t}e^{-i\hat{\omega}_{m-1}n}.$$

5. SIMULATION RESULTS

In order to examine the behaviour of the statistics

$$T_1 = N^{3/2}(\hat{\omega}_s^{(7)} - \omega_s)$$

$$T_2 = N^{1/2}(re\ \hat{\alpha}_s - re\ \alpha_s)$$

$$T_3 = N^{1/2}(im\ \hat{\alpha}_s - im\ \alpha_s)$$

for $s = 1,2,3$, as N increases, the following simulation study was carried out.

Using the model

$$Y_t = \alpha_1 e^{i\omega_1 t} + \alpha_2 e^{i\omega_2 t} + \alpha_3 e^{i\omega_3 t} + \epsilon_t$$
$$t = 1,\ldots,N$$

for 3 signals with the true values

$$\omega_1 = 1.5, \quad \omega_2 = 2.1, \quad \omega_3 = 2.9$$

$$re\ \alpha_i = 5.2, \quad im\ \alpha_i = 0, \quad i = 1,2,3$$

$$\sigma^2 = 5.0$$

independent samples of sizes varying from 50 to 2500 at intervals of 50 were drawn and the estimates of all the parameters were computed as described in Section 4.

Simulations were done using the two methods given in (4.1) and (4.11). The results show:

(i) When the sample size n is greater than 300, there is not much difference between the two methods. In this case (4.1) provides a simple method.

(ii) When the sample size is less than 300, the results by (4.1) appear to be less stable than those by (4.11). Even when n is as small as 50, the results by (4.11) show considerable improvement over the EVLP method.

The graphs of the statistics T_1, T_2 and T_3, obtained by the method (4.11), against the sample size N are shown in Figures 1, 2, and 3 respectively. It is seen from the graphs that stability is reached at a sample size of the order of 250.

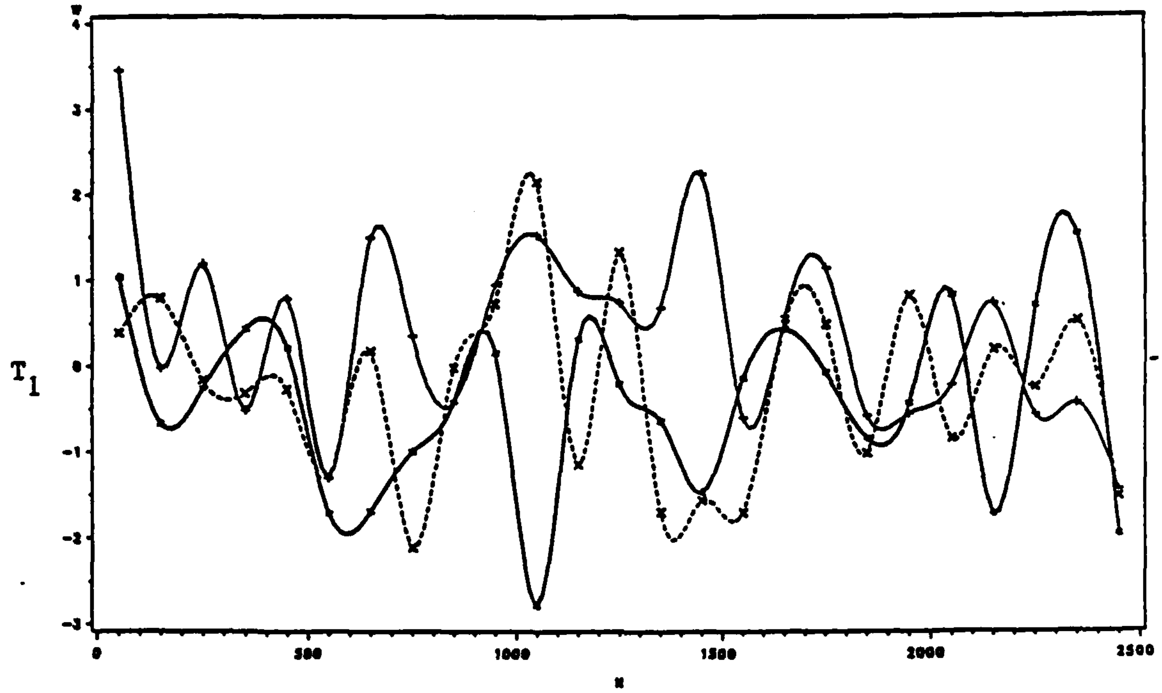

Figure 1. Graph of T_1 against sample size N

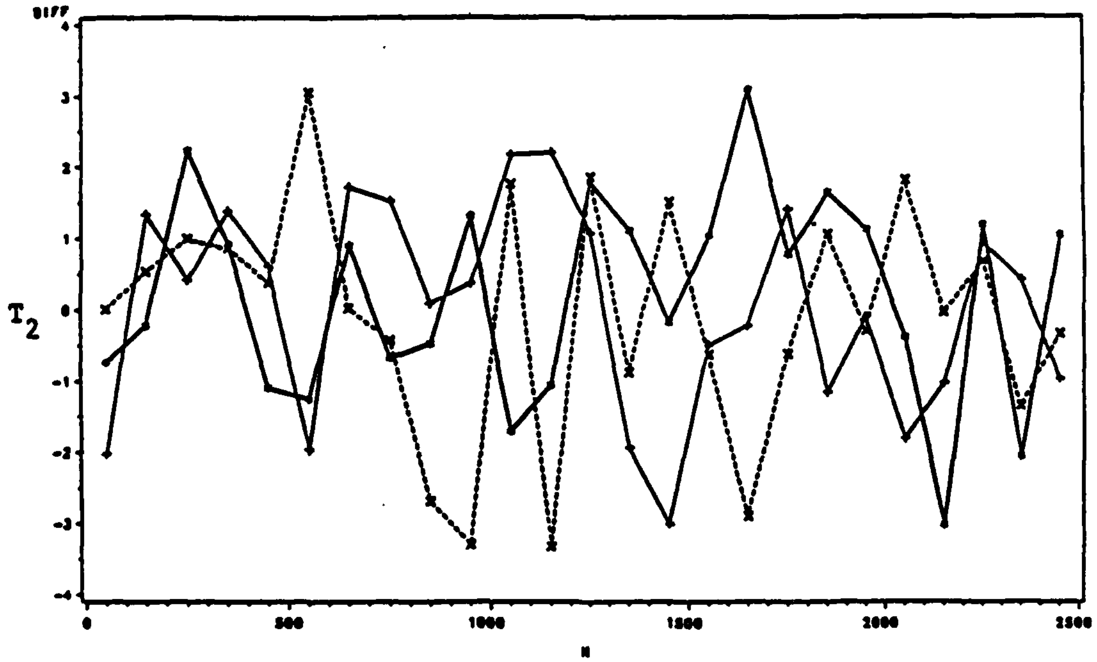

Figure 2. Graph of T_2 against sample size N

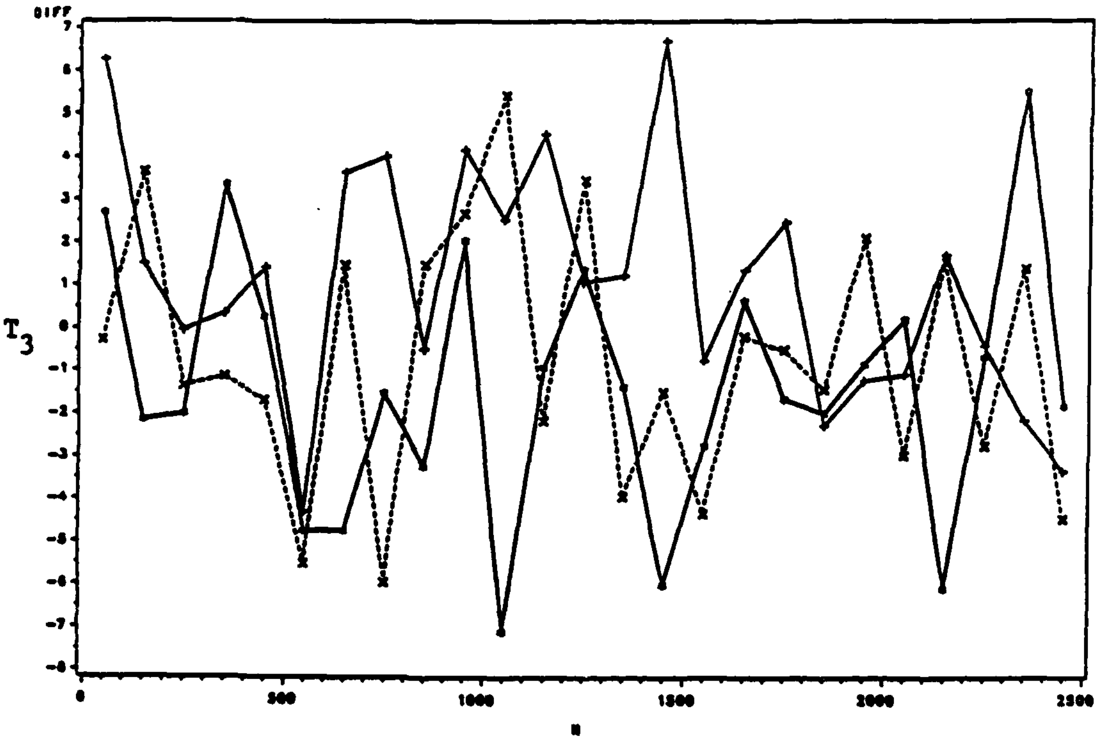

Figure 3. Graph of T_3 against sample size N

The estimates of ω_1, ω_2, ω_3 obtained after 7 steps of refinement over the EVLP estimates are given in the following Table. The estimates are close to the true values even for small sample sizes.

TABLE. Estimates of ω_1, ω_2, ω_3 obtained after 7 steps of refinement over the EVLP estimates.

N	$\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{\omega}_3$
50	1.4902339	2.0988951	2.8970912
150	1.5000117	2.0995676	2.9003642
250	1.4997012	2.1000414	2.9000647
350	1.5000774	2.1000476	2.8999323
450	1.4999175	2.1000282	2.8999789
550	1.5001009	2.1000981	2.9001332
650	1.4999101	2.0999894	2.9001015
750	1.4999827	2.1001022	2.9000483
850	1.5000169	2.1000005	2.9000164
950	1.4999677	2.0999751	2.8999948
1050	1.4999554	2.0999370	2.9000816
1150	1.4999777	2.1000294	2.8999919
1250	1.4999834	2.0999702	2.9000048
1350	1.4999863	2.1000342	2.9000128
1450	1.4999595	2.1000282	2.9000266
1550	1.5000098	2.1000278	2.9000022
1650	1.4999914	2.0999927	2.8999936
1750	1.4999845	2.0999933	2.9000010
1850	1.5000072	2.1000128	2.9000106
1950	1.5000064	2.0999905	2.9000050
2050	1.5000024	2.1000091	2.8999912
2150	1.4999927	2.0999981	2.9000172
2250	1.5000053	2.1000023	2.8999934
2350	1.5000038	2.0999954	2.8999866
2450	1.5000119	2.1000125	2.9000161

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A computational algorithm is given for obtaining asymptotically efficient estimates of the unknown complex amplitudes and frequencies in a superimposed exponential model for signals. It is shown that the variance covariance matrix of these estimates are asymptotically same as that for the maximum likelihood estimates and thus attain the Cramér-Rao lower bound.